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ON THE QUASI-STEADINESS HYPOTHESIS AS APPLIED TO GAS EXHAUSTION FROM A RECEIVER

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A semi-empirical method of determining the stabilization time for a quasi-steady mode of gas exhaustion from a receiver after sudden opening of the nozzle and the time evolution of the real flow rate at the stage of the transitional process are considered. The numerical solution of the equations of exhaustion gas dynamics in a two-dimensional formulation and the results of model experiments demonstrated that the method can be used to estimate the conditions of applicability of the quasi-steadiness hypothesis and to determine the discharge coefficient of the nozzle with controlled accuracy.

Key words: *receiver, flow rate, quasi-steady mode.*

One of the central assumptions in mathematical modeling of transitional processes in a semi-closed volume within the framework of the thermodynamic (zero-dimensional) approach is the hypothesis of exhaustion quasi-steadiness [1]. According to this hypothesis, no matter how rapidly the governing parameters (e.g., the nozzle-throat area $F_{\rm th}$) change, the mass flow rate of the gas through the nozzle per second corresponds to instantaneous values of these parameters and is determined by the quasi-steady dependence on time, which has the following form for the supercritical flow mode:

$$G_{\rm qst}(t) = \varphi_0 p(t) F_{\rm th}(t) \Gamma(k)/\sqrt{\chi R T(t)}. \tag{1}$$

Here $p(t)$ and $T(t)$ are the current values of stagnation pressure and temperature in the receiver (averaged over the free volume of the receiver V), $\Gamma(k) = \sqrt{k}(2/(k+1))^{(k+1)/(2(k-1))}$, k is the ratio of specific heats, χ is the coefficient of heat losses (in what follows, it is assumed that $\chi = 1$, i.e., the receiver walls are thermally insulated), R is the gas constant of exhaustion products, and φ_0 is the discharge coefficient of the nozzle.

The physical meaning of the hypothesis is that the velocity of downstream propagation of disturbances is significantly greater than the rate of variation of parameters at an arbitrary point of the flow. The use of this hypothesis appreciably simplifies problem formulation and interpretation of results in theoretical and experimental investigations of transitional processes in various engineering devices. The practical importance of the hypothesis, however, is determined by careful justification of the conditions and limits of its applicability. These issues have been studied most extensively as applied to shock starting of nozzles and processes in hotshot wind tunnels [2–7]. Methods of numerical calculation of a one- or two-dimensional unsteady flow generated by a sudden breakdown of the membrane separating the receiver from the nozzle or from the ambient medium were used in most papers. The initial stage of exhaustion from an instantaneously opened nozzle is accompanied by complicated wave phenomena, which were analyzed in detail in [6]. The unsteady disturbances decay with time, which finally ensures a quasi-steady mode of exhaustion.

Note, in the papers mentioned above, the calculations were performed for particular values of regime and geometric parameters of the problem; the values of exhaustion quasi-steadiness stabilization times refer to the conditions considered. The issues of comparison of unsteady and quasi-steady approaches are described in much detail in [5, 7]. Zvegintsev and Shashkin [7] estimated the difference in pressures in the receiver, which were

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calculated for both problem formulations with exhaustion from a cylindrical receiver with a cross-sectional area $F_{\rm ch}$ through a nozzle with a throat area $F_{\rm th}$, which was mounted in one of the butt-end faces of the receiver. For small values of the parameter $\alpha = F_{\rm th}/F_{\rm ch}$, we have

$$\varepsilon_P = \frac{p_{\rm unst} - p_{\rm qst}}{p_{\rm qst}} = \Big[1 + \beta_1 \frac{\alpha^2 \bar{t}}{1 + \beta_2^2 \bar{t}}\Big]^{-2k/(k-1)} - 1,$$

where

$$\beta_1 = \frac{(k+1)(k-1)}{4}\Big(\frac{2}{k+1}\Big)^{3(k+1)/(2(k-1))}, \qquad \beta_2 = \frac{k-1}{2}\Big(\frac{2}{k+1}\Big)^{(k+1)/(2(k-1))},$$

$\bar{t} = t/B$, $B = V/(a_0 F_{\rm th})$, and a_0 is the velocity of sound based on the initial values of gas parameters in the receiver.

Based on numerical calculations of a one-dimensional unsteady flow in a hotshot wind tunnel, Zvegintsev and Shashkin [7] analyzed the conditions of applicability of the quasi-steadiness hypothesis for various nozzle geometries. It was shown that the process of nozzle starting includes propagation of the first rarefaction wave in the chamber during the time t_1 and a system of shock waves in the nozzle during the time t_2. The run duration is determined by the greater time (t_1 or t_2). To approximately estimate the time of nozzle starting $t_{\rm n}$, it was suggested to use the expressions

$$t_1 = 2L_{\rm ch}/a_0, \qquad t_2 = 2L_{\rm n}/a_0,$$

where $L_{\rm ch}$ and $L_{\rm n}$ are the chamber and nozzle lengths, respectively.

The estimates obtained by the above-given formulas show that the quasi-steady flow stabilization time is $t \approx 10^{-2}$–10^{-4} sec, i.e., the applicability of the quasi-steadiness hypothesis is validated for most problems of practical importance associated with investigations of transitional processes.

In some cases, however, it is necessary to take into account exhaustion unsteadiness, if the characteristic times of internal processes in the chamber are commensurable with $t_{\rm n}$. As an example, we can mention problems associated with unsteady combustion of gases or condensed substances in a semi-closed volume when the characteristic times of chemical reactions are small enough [8]. In this case, one should know both the time of stabilization of the quasi-steady exhaustion mode and the time evolution of the flow rate in transitional processes (e.g., in the case of pressure release in a rocket engine). Numerical computations of such problems should be normally performed in a two-dimensional formulation with allowance for particular geometry with a rather fine grid.

In the present paper, we consider a semi-empirical method for determining the stabilization time of the quasi-steady mode of gas exhaustion from the receiver after sudden opening of the nozzle and also for determining the real flow rate $G(t)$ at the stage of the transitional process. In addition, this method allows one to determine the discharge coefficient of the nozzle with controlled accuracy. The method is based on measuring the time evolution of pressure $p(t)$ in the case of adiabatic exhaustion from the receiver through the nozzle.

The system of equations in variables averaged over the receiver volume has the form

$$V\,\frac{d\rho}{dt} = -G, \qquad \frac{p}{\rho^k} = {\rm const}, \qquad p = \rho RT, \tag{2}$$

where ρ is the gas density in the receiver.

The initial conditions are $G = 0$, $p = p_0$, and $T = T_0$.

We denote the ratio of the real flow rate to the quasi-steady value determined by Eq. (1) as $g(t)$:

$$g(t) = G(t)/G_{\rm qst}(t).$$

In the course of exhaustion, the value of $g(t)$ changes and reaches $g(t) = 1$ when the quasi-steady mode is reached. The first equation of system (2) acquires the form

$$V\,\frac{d\rho}{dt} = -\varphi_0 g p F_{\rm th}\,\frac{\Gamma(k)}{\sqrt{RT}}. \tag{3}$$

The quantity $\varphi(t) \equiv \varphi_0 g(t)$ in the quasi-steady exhaustion mode equals the discharge coefficient of the nozzle φ_0. In the case of adiabatic discharge from a semi-closed volume through the nozzle, Eq. (3) with allowance for (2) can be presented in the form

$$\frac{d}{dt}\Big(\frac{p}{p_0}\Big) = -\varphi\,\frac{k}{t_{\rm ch}}\Big(\frac{p}{p_0}\Big)^{(3k-1)/(2k)},$$

where $t_{\text{ch}} = V/(F_{\text{th}}\Gamma(k)\sqrt{RT_0})$ is the characteristic exhaustion time. This equation yields the expression for determining $\varphi(t)$:

$$\varphi(t) = -\frac{t_{\text{ch}}}{k}\Big(\frac{p(t)}{p_0}\Big)^{(1-3k)/(2k)} \frac{d}{dt}\Big(\frac{p(t)}{p_0}\Big). \tag{4}$$

Here $p(t)$ is the experimentally measured dependence of pressure in the receiver on time.

We present Eq. (4) in the form

$$\varphi(t) = \frac{d}{dt} f(t),$$

where

$$f(t) = \frac{2}{k-1}\, t_{\text{ch}}\Big[\Big(\frac{p_0}{p(t)}\Big)^{(k-1)/(2k)} - 1\Big]. \tag{5}$$

The problem of determining $\varphi(t)$ reduces to finding the derivative of the experimental function (5). Numerical differentiation of experimental functions is an ill-posed problem, and smoothing has to be performed before solving this problem. This is done either by a smoothing spline or by a function whose form is chosen on the basis of *a priori* information on properties of the problem solution; the unknown parameters are chosen by the least squares technique. To determine the form of the functional dependence $\varphi(t)$, we supplement system (2) by the equation

$$\frac{dG(t)}{dt} = -\frac{G(t) - G_{\text{qst}}(t)}{t_{\text{n}}}, \tag{6}$$

where t_{n} is the characteristic time of stabilization of the quasi-steady exhaustion mode.

We introduce the dimensionless variables

$$\tau = t/t_{\text{ch}}, \qquad x = p/p_0, \qquad y = G/G_0,$$

where G_0 is determined by formula (1): $G_0 = \varphi_0 p_0 F_{\text{th}}\Gamma(k)/\sqrt{RT_0}$. In the dimensionless form, system (2) with allowance for (6) reduces to

$$\frac{dx}{d\tau} = -kx^{(k-1)/k}y, \qquad \varepsilon\,\frac{dy}{d\tau} = x - x^{(k-1)/(2k)}y, \tag{7}$$

where $\varepsilon = t_{\text{n}}/t_{\text{ch}}$ is a small parameter.

The initial conditions are $x(0) = 1$ and $y(0) = y^0$.

System (7) is singularly disturbed. To solve this system, we use the method of composite expansions of the boundary functions [9]; the solution has the form

$$x(\tau,\varepsilon) = X(\tau,\varepsilon) + u(\xi,\varepsilon), \qquad y(\tau,\varepsilon) = Y(\tau,\varepsilon) + v(\xi,\varepsilon),$$

where $\xi = \tau/\varepsilon = t/t_{\text{n}}$, X and Y refer to the external solution and u and v refer to the internal solution. The asymptotic expansions for X, Y, u, and v are sought as the series

$$X(\tau,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n X_n(\tau), \qquad Y(\tau,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n Y_n(\tau), \qquad u(\xi,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n u_n(\xi), \qquad v(\xi,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n v_n(\xi).$$

In seeking the solution, we confine ourselves to the main term of the asymptotic expansion for X and Y and two first terms of the asymptotic series for u and v. The regular part of the solution (corresponding to the quasi-steady mode) is written as

$$X_0 = (1 + (k-1)\tau/2)^{-2k/(k-1)}, \qquad Y_0 = (1 + (k-1)\tau/2)^{-(k+1)/(k-1)},$$

and the internal solution is

$$u(\xi,\varepsilon) = u_0 + u_1\varepsilon = \varepsilon k(1 - y^0)(1 - \mathrm{e}^{-\xi}), \qquad v(\xi,\varepsilon) = v_0 + v_1\varepsilon = -(1 - y^0)\,\mathrm{e}^{-\xi}\,.$$

Thus, we have

$$\begin{aligned} x(\tau,\varepsilon) &= (1 + (k-1)\tau/2)^{-2k/(k-1)} + \varepsilon k(1 - y^0)(1 - \mathrm{e}^{-\tau/\varepsilon}), \\ y(\tau,\varepsilon) &= (1 + (k-1)\tau/2)^{-(k+1)/(k-1)} - (1 - y^0)\,\mathrm{e}^{-\tau/\varepsilon}\,. \end{aligned} \tag{8}$$

In the dimensional form, the second equation of system (8) for $y^0 = 0$ (instantaneous opening of the nozzle) becomes

$$G(t) = G_0\Big[\Big(1 + \frac{k-1}{2}\,\frac{t}{t_{\rm ch}}\Big)^{-(k+1)/(k-1)} - {\rm e}^{-t/t_{\rm n}}\Big],$$

and the function $\varphi(t)$ can be presented in the form

$$\varphi(t) = \varphi_0\Big[1 - \Big(1 + \frac{k-1}{2}\,\frac{t}{t_{\rm ch}}\Big)^{(k+1)/(k-1)}\,{\rm e}^{-t/t_{\rm n}}\Big].$$

Stabilization of the quasi-steady exhaustion mode proceeds during the characteristic time $t_{\rm n}$; therefore (with allowance that $t_{\rm n} \ll t_{\rm ch}$), we obtain

$$\varphi(t) = \varphi_0[1 - {\rm e}^{-t/t_{\rm n}}]. \tag{9}$$

At the moment of nozzle opening, the discharge coefficient of the gas is assumed to be zero $\varphi(0) = 0$. Then, the discharge coefficient monotonically increases to its steady value φ_0 with the characteristic time $t_{\rm n}$. Note, for $\varphi_0 = {\rm const}$ and adiabatic exhaustion, the function $f(t)$ is linear. Approximation (9) for $\varphi(t)$ means that the experimental dependence $f(t)$ is approximated by the function

$$f(t) = \int\limits_0^t \varphi(t)\,dt = \varphi_0\{t - t_{\rm n}[1 - {\rm e}^{-t/t_{\rm n}}]\}. \tag{10}$$

To find φ_0 and $t_{\rm n}$, we use the least squares technique, which minimizes the functional

$$S = \sum_{i=1}^{n}\Big[f_i - \varphi_0\{t_i - t_{\rm n}[1 - {\rm e}^{-t/t_{\rm n}}]\}\Big]^2,$$

where $f(t_i)$ are calculated by formula (10) for the measured values of p_i. Dependence (10) is nonlinear with respect to the parameter $t_{\rm n}$; therefore, iterations with respect to this parameters were performed until convergence was reached.

For $t_{\rm n} \ll t_{\rm ch}$, the function $f(t)$ is approximated by a straight line, which allows us to simplify the procedure of finding the discharge coefficient. It is sufficient to use two points on the pressure-release curve: at the beginning (t_1, p_1) and at the end (t_2, p_2) of the process. For a linear dependence $f(t)$, the tangent of its angle of inclination is found as the first difference:

$$\varphi_0 = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{t_{\rm ch}}{t_2 - t_1}\,\frac{2}{k-1}\Big[\Big(\frac{p_0}{p_2}\Big)^{(k-1)/(2k)} - \Big(\frac{p_0}{p_1}\Big)^{(k-1)/(2k)}\Big].$$

The accuracy of determining φ_0 and $t_{\rm n}$ by the method considered depends on the measurement error of the initial parameters (pressure in the receiver) and on the error of approximation of the derivative by the finite difference. We can show that the relative error of determining the discharge coefficient $\delta\varphi_0$ is related to the absolute error of pressure measurement Δp by the formula

$$\delta\varphi_0 \approx \max\Big(\frac{k-1}{2k}\,\frac{\Delta p}{p_2}, \Big|2\,\frac{f_1 - 2\langle f\rangle + f_2}{f_2 - f_1}\Big|\Big),$$

where f_1, $\langle f\rangle$, and f_2 are determined for the times t_1, $\langle t\rangle = (t_1 + t_2)/2$, and t_2. In choosing the value of $p_2 = (k-1)p_0/(2k)$, the relative measurement error of the discharge coefficient $\delta\varphi_0$ coincides with the relative error of pressure measurement δp_0, which can be, apparently, considered as the optimal condition. Since the value of $t_{\rm n}$ is determined by the inflection point on the curve $f(t)$, the error of determining the characteristic time $t_{\rm n}$ also corresponds to the error of pressure measurement. If standard transducers are used (e.g., of the LKh series), the error of determining $t_{\rm n}$ is within 5–10%.

Dependence (9) was verified numerically. An unsteady axisymmetric flow of an inviscid heat-non-conducting gas with constant specific heats in the computational domain of the experimental setup (Fig. 1) after sudden opening of the section FF_1 was considered.

The laws of conservation of mass, momentum, and energy in the integral form are

$$\frac{d}{dt}\iint\limits_S \rho r\,dz\,dr + \oint\limits_C \rho r(u_z\,dr - u_r\,dz) = 0,$$

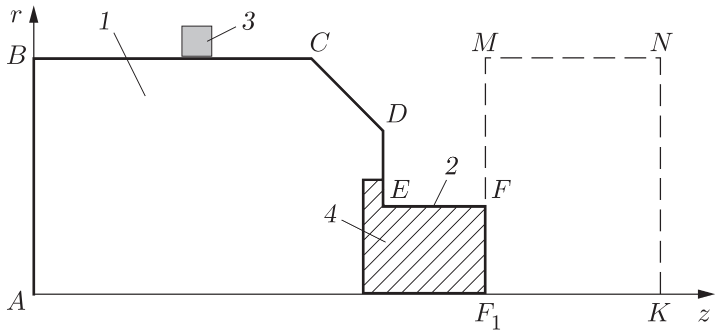

Fig. 1. Layout of the experimental setup and computational domain: 1) receiver; 2) nozzle; 3) pressure transducer; 4) burndown plug.

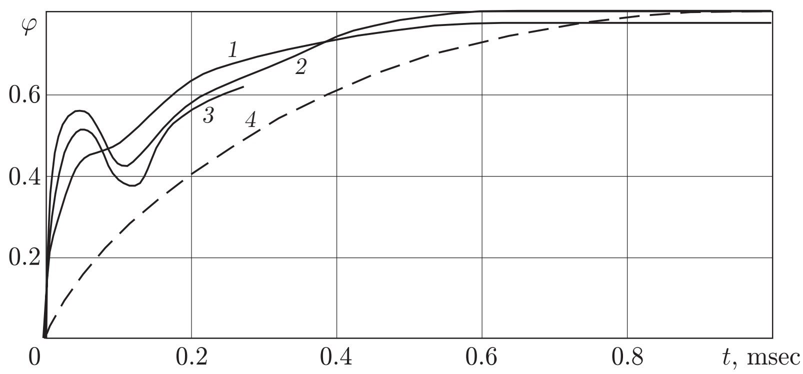

Fig. 2. Stabilization of the quasi-steady exhaustion mode: curves 1–3 show the results of numerical calculations for 16×4 grid (1), 64×16 grid (2), and 256×64 grid (3); curve 4 shows the data obtained by the semi-empirical method ($d_{\rm th} = 15.2$ mm, $p_0 = 4.84$ MPa, and $\langle dp/dt\rangle = 2\cdot10^2$ MPa/sec).

$$\frac{d}{dt}\iint\limits_S \rho u_z r\,dz\,dr + \oint\limits_C r[(p+\rho u_z^2)\,dr - \rho u_r u_z\,dz] = 0,$$

$$\frac{d}{dt}\iint\limits_S \rho u_r r\,dz\,dr + \oint\limits_C r[\rho u_z u_r\,dr - (p+\rho u_r^2)\,dz] = \iint\limits_S p\,dz\,dr,$$

$$\frac{d}{dt}\iint\limits_S \rho(2e+q^2) r\,dz\,dr + \oint\limits_C r\{[2pu_z + \rho u_z(2e+q^2)]\,dr - [2pu_r + \rho u_r(2e+q^2)]\,dz\} = 0.$$

Here z and r are the axial and radial coordinates, u_z and u_r are the components of the velocity vector of the gas along z and r, respectively, $q = \sqrt{u_r^2+u_z^2}$, $e = p/((k-1)\rho)$ is the specific internal energy, and C is an arbitrary closed contour bounding the area S.

The boundary conditions were the no-slip condition $\boldsymbol{v}_n = 0$ on the receiver walls ABCDEF and on the axis of symmetry; the boundary condition on the line FMNK was the "soft" condition $\partial\boldsymbol{v}_n/\partial n = 0$, which does not affect the flow in the receiver because $p_0 \gg p_a$, where p_a is the ambient pressure. At the initial time ($t = 0$), the gas inside and outside the receiver is at rest [$u_r(0) = u_z(0) = 0$], and the values of p_0 and p_a are prescribed.

To solve the problem, we used the first-order numerical integration, which is based on the idea of using exact solutions of equations with piecewise-constant data for constructing the difference scheme [10].

Figure 2 shows the results of numerical calculations of quasi-steady exhaustion mode stabilization with different grids in the z and r directions (16×4, 64×16, and 256×64) for actual conditions of model experiments: nozzle-throat diameter $d_{\rm th} = 15.2$ mm, initial pressure in the receiver $p_0 = 4.84$ MPa, and mean gradient of pressure release $\langle dp/dt\rangle = 2\cdot10^2$ MPa/sec. The figure also shows the dependence $\varphi(t)$ obtained by processing the experimental curve $p(t)$ by the semi-empirical method considered.

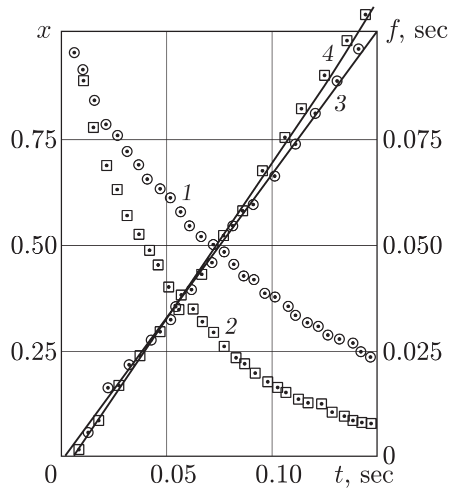

Fig. 3. Experimental curves of pressure release in the receiver: $x(t)$ for $d_{\rm th} = 5$ (1) and 7 mm (2); $f(t)$ for $d_{\rm th} = 5$ (3) and 7 mm (4).

Two setups were used for experimental validation of the method proposed. The layout of the first setup is shown in Fig. 1. The nozzle was opened when the burndown plug left the nozzle through the cross section FF_1 (see Fig. 1) under the action of the pressure in the receiver. The plug material was the double-base propellant N whose combustion products generated a given pressure in the receiver p_0 and escaped through the nozzle after the plug left the latter. The time of nozzle opening Δt was estimated by the formula derived by integration of the equation of plug motion (without allowance for the friction force)

$$\Delta t = \sqrt{2\rho_m l/p_0}\,(\sqrt{l + d_{\rm th}/4} - \sqrt{l}\,),$$

where ρ_m is the powder density and l is the plug length.

For conditions of the tests performed, we have $\Delta t \approx 0.45 \cdot 10^{-4}$ sec, which is an order of magnitude lower than the time needed for stabilization of the quasi-steady exhaustion mode.

A comparison of results obtained by numerical calculations and by the semi-empirical method (Fig. 2) shows that the method considered yields a fairly adequate approximation of the real flow rate $G(t)$ in the course of stabilization of the quasi-steady exhaustion mode, whereas the time of quasi-steady mode stabilization is somewhat different (0.5–0.6 msec in the numerical calculations and 0.8–0.9 msec predicted by the semi-empirical method). The characteristic time in Eq. (9) is $t_{\rm n} = 0.29$ msec.

The second setup is designed for determining the discharge coefficient of nozzles with different configurations. It consists of an array of compressed gas holders, receiver, nozzle under study, electric pneumatic valve, and system for measuring the pressure in the receiver. Compressed air is injected into the receiver up to a prescribed pressure p_0; when the valve is opened, the air from the receiver escapes through the nozzle into the atmosphere. The pressure-release curve is registered by a transducer with the data recorded onto a storage oscillograph.

This setup does not allow determination of the stabilization time $t_{\rm n}$ because of the nozzle-opening inertia, but it is simple in operation and more convenient for serial tests.

Figure 3 shows the curves $x(t) = p(t)/p_0$ for two nozzles with abrupt constriction of the flow; the nozzle-throat diameters are 5.0 and 7.0 mm. The relative cross-sectional areas of the nozzle (nozzle unit) are $F_{\rm th}/F_{\rm ch} = 0.01$ and 0.02, respectively. The same figure shows the functions $f(t)$ for these nozzles. The scatter of points characterizes the error of pressure measurement (approximately 2%). The tangent of the angle of inclination of the asymptotic curves $f(t)$ is correlated with the discharge coefficient φ_0. Processing of experimental data by the technique described above yields the discharge coefficients $\varphi_0 = 0.70 \pm 0.04$ ($d_{\rm th} = 5.0$ mm) and $\varphi_0 = 0.73 \pm 0.03$ ($d_{\rm th} = 7.0$ mm). The calculations by the approximate formulas [10] yield the discharge coefficients $\varphi_0 = 0.738$ and $\varphi_0 = 0.739$, respectively. The estimate for the stabilization time of the quasi-steady exhaustion mode for the conditions of the tests performed ($t_{\rm ch} \approx 10^{-2}$ sec) shows that $t_{\rm n} \ll t_{\rm ch}$. Note, to obtain a quantitative estimate for $t_{\rm n}$ by the method considered, one has to use pressure transducers with eigenfrequencies above 50–100 kHz.

Let us summarize the results of the present study.

1. An exponential dependence of the real flow rate of the gas from the receiver with abrupt opening of the nozzle in the period of stabilization of the quasi-steady exhaustion mode was obtained by the method of asymptotic expansions of solutions of singularly disturbed equations.

2. The adequacy of the resultant flow-rate dependence was obtained by numerically solving two-dimensional gas-dynamic equations.

3. A semi-empirical method was proposed for determining the time of stabilization of quasi-steady exhaustion from the receiver and the discharge coefficient of the nozzle with controlled accuracy.

4. The results of experiments on exhaustion of compressed air and combustion products of the powder N from the receiver with sudden opening of the nozzle confirmed the possibility of using the method for practical evaluation of the conditions of applicability of the quasi-steadiness hypothesis and for determining the discharge coefficient of a nozzle with an arbitrary configuration.

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